Lieb-Thirring inequalities with improved constants

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Abstract

Following Eden and Foias we obtain a matrix version of a generalised Sobolev inequality in one-dimension. This allows us to improve on the known estimates of best constants in Lieb-Thirring inequalities for the sum of the negative eigenvalues for multi-dimensional Schrödinger operators.

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1 Introduction

Let H be a Schrödinger operator in $L^2(\mathbb{R}^d)$

$$H = -\Delta - V \tag{1}$$

For a real-valued potential V we consider Lieb-Thirring inequalities for the negative eigenvalues $\{\lambda_n\}$ of the operator H

$$\sum |\lambda_n|^\gamma \leq L_{d,\gamma} \int_{\mathbb{R}^d} V_+^{d/2+\gamma}(x)\, dx\,, \tag{2}$$

where $V_+ = (|V| + V)/2$ is the positive part of V.

Eden and Foias have obtained in [3] a version of a one-dimensional generalised Sobolev inequality which gives best known estimates for the constants in the inequality (2) for $1 \leq \gamma < 3/2$. The aim of this short article is to extend the method from [3] to a class of matrix-valued potentials. By using ideas from [6] this automatically improves on the known estimates of best constants in (2) for multidimensional Schrödinger operators.

Lieb-Thirring inequalities for matrix-valued potentials for the value $\gamma = 3/2$ were obtained in [6] and also in [2]. Here we state a result corresponding to $\gamma = 1$.

Theorem 1. *Let $V \geq 0$ be a Hermitian $m \times m$ matrix-function defined on $\mathbb{R}$ and let λ_n be all negative eigenvalues of the operator* (1)*. Then*

$$\sum |\lambda_n| \leq \frac{2}{3\sqrt{3}} \int_{\mathbb{R}} \mathrm{Tr}\, \left[V^{3/2}(x)\right]\, dx\,. \tag{3}$$

Remark 1. *The constant* $\frac{2}{3\sqrt{3}}$ *should be compared with the Lieb-Thirring constant found in [7] for a class of single eigenvalue potentials and with the constant obtained in [5] which is twice as large as the semi-classical one*

$$\frac{4}{3\sqrt{3}\,\pi} < \frac{2}{3\sqrt{3}} < 2 \times \frac{2}{3\pi} = 2 \times \frac{1}{2\pi} \int_{\mathbb{R}} (1 - \xi^2)_+ \, d\xi \, .$$

This is about $0,2450\cdots < 0,3849\cdots < 0,4244\ldots$.

Remark 2. *Note that the values of the best constants for the range* $1/2 < \gamma < 3/2$ *remain unknown.*

Let $\mathcal{A}(x) = (a_1(x), \ldots, a_d(x))$ be a magnetic vector potential with real valued entries $a_k \in L^2_{\text{loc}}(\mathbb{R}^d)$ and let

$$H(\mathcal{A}) = (i\,\nabla + \mathcal{A})^2 - V \, ,$$

where $V \geq 0$ is a real-valued function.

Denote the ratio of $2/3\sqrt{3}$ and the semi-classical constant by

$$R := \frac{2}{3\sqrt{3}} \times \left(\frac{2}{3\pi}\right)^{-1} = 1.8138\ldots.$$

By using the Aizenmann-Lieb argument [1], a "lifting" with respect to dimension [6], [5], and Theorem 1 we obtain the following result:

Theorem 2. *For any* $\gamma \geq 1$ *and any dimension* $d \geq 1$*, the negative eigenvalues of the operator* $H(\mathcal{A})$ *satisfy inequalities*

$$\sum |\lambda_n|^\gamma \leq L_{d,\gamma} \int_{\mathbb{R}^d} V^{d/2+\gamma}(x) \, dx \, ,$$

where

$$L_{d,\gamma} \leq R \times L^{cl}_{d,\gamma} = R \times \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1 - |\xi|)^\gamma_+ \, d\xi \, .$$

Remark 3. *Theorem 2 allows us to improve on the estimates of best constants in Lieb-Thirring inequalities for Schrödinger operators with complex-valued potentials recently obtained in [4].*

2 One-dimensional generalised Sobolev inequality for matrices

Let $\{\phi_n\}_{n=0}^N$ be an ortho-normal system of vector-functions in $L^2(\mathbb{R}, \mathbb{C}^M)$, $M \in \mathbb{N}$,

$$(\phi_n, \phi_m) = (\phi_n, \phi_m)_{L^2(\mathbb{R},\mathbb{C}^M)} = \sum_{j=1}^{M} \int_{\mathbb{R}} \phi_n(x,j) \, \overline{\phi_m(x,j)} \, dx = \delta_{nm} \, ,$$

where δ_{nm} is the Kronecker symbol. Let us introduce an $M \times M$ matrix U with entries

$$u_{j,k}(x,y) = \sum_{n=0}^{N} \phi_n(x,j)\,\overline{\phi_n(y,k)}\,.$$

Clearly

$$U^*(x,y) = U(y,x)\,. \tag{4}$$

The fact that the functions ϕ_n are orthonormal can be written in a compact form

$$\int_{\mathbb{R}} U(x,y)\,U(y,z)\,dy = U(x,y)\,. \tag{5}$$

The latter two properties (4) and (5) prove that $U(x,y)$ could be considered as the integral kernel of an orthogonal projection P in $L^2(\mathbb{R},\mathbb{C}^M)$ whose image is the subspace of vector-functions spanned by $\{\phi_n\}_{n=1}^N$.

Theorem 3. *Let us assume that the vector-function ϕ_n, $n = 1, 2, \ldots N$, are from the Sobolev class $H^1(\mathbb{R},\mathbb{C}^M)$. Then*

$$\int_{\mathbb{R}} \mathrm{Tr}\left[U(x,x)^3\right] dx \le \sum_{n=1}^{N}\sum_{j=1}^{M} \int_{\mathbb{R}} |\phi_n'(x,j)|^2\,dx\,.$$

Proof.

$$\begin{aligned}
&\frac{d}{dy}\,\mathrm{Tr}\left[U(x,y)\,U(y,x)\,U(x,x)\right] \\
&= \mathrm{Tr}\left[\Big(\frac{d}{dy}\,U(x,y)\Big)\,U(y,x)\,U(x,x)\right] + \mathrm{Tr}\left[U(x,y)\,\Big(\frac{d}{dy}\,U(y,x)\Big)\,U(x,x)\right]
\end{aligned} \tag{6}$$

By integrating (6) and taking absolute values one obtains

$$\begin{aligned}
\frac{1}{2}\,\mathrm{Tr}&\left[U(x,z)\,U(z,x)\,U(x,x)\right] \\
&\le \frac{1}{2}\int_{-\infty}^{z} \Big|\,\mathrm{Tr}\left[\Big(\frac{d}{dy}\,U(x,y)\Big)\,U(y,x)\,U(x,x)\right] \\
&\qquad + \mathrm{Tr}\left[U(x,y)\,\Big(\frac{d}{dy}\,U(y,x)\Big)\,U(x,x)\right]\Big|\,dy
\end{aligned}$$

and

$$\frac{1}{2}\,\mathrm{Tr}\,\Big[U(x,z)\,U(z,x)\,U(x,x)\Big] \leq \frac{1}{2}\int_z^\infty \Big|\,\mathrm{Tr}\,\Big[\Big(\frac{d}{dy}\,U(x,y)\Big)\,U(y,x)\,U(x,x)\Big] + \mathrm{Tr}\,\Big[U(x,y)\,\Big(\frac{d}{dy}\,U(y,x)\Big)\,U(x,x)\Big]\Big|\,dy\,.$$

Taking absolute values and adding the two inequalities yields for any $z \in \mathbb{R}$

$$\Big|\,\mathrm{Tr}\,\Big[U(x,z)\,U(z,x)\,U(x,x)\Big]\Big| \leq \frac{1}{2}\int_{\mathbb{R}}\Big|\,\mathrm{Tr}\,\Big[\Big(\frac{d}{dy}\,U(x,y)\Big)\,U(y,x)\,U(x,x)\Big]\Big|\,dy + \frac{1}{2}\int_{\mathbb{R}}\Big|\,\mathrm{Tr}\,\Big[U(x,y)\,\Big(\frac{d}{dy}\,U(y,x)\Big)\,U(x,x)\Big]\Big|\,dy\,. \quad (7)$$

Note that we have reproved Agmon's inequality

$$|f(x)|^2 \leq \int_{\mathbb{R}} |f(y)\,f'(y)|\,dy$$

for traces of matrices. By using properties of traces, the Cauchy-Schwarz inequality for matrix-functions and also properties (4) and (5), we find that for all $z \in \mathbb{R}$

$$\begin{aligned}
&\Big(\int_{\mathbb{R}}\Big|\,\mathrm{Tr}\,\Big[\Big(\frac{d}{dy}\,U(x,y)\Big)\,U(y,x)\,U(x,x)\Big]\Big|\,dy\Big)^2 \\
&\leq \int_{\mathbb{R}} \mathrm{Tr}\,\Big[\frac{d}{dy}\,U(x,y)^*\,\frac{d}{dy}\,U(x,y)\Big]\,dy\,\int_{\mathbb{R}} \mathrm{Tr}\,\big[U(x,y)^*\,U^2(x,x)\,U(x,y)\big]\,dy \\
&= \int_{\mathbb{R}} \mathrm{Tr}\,\Big[\frac{d}{dy}\,U(y,x)\,\frac{d}{dy}\,U(x,y)\Big]\,dy\,\int_{\mathbb{R}} \mathrm{Tr}\,\big[U^2(x,x)\,U(x,y)\,U(y,x)\big]\,dy \\
&= \int_{\mathbb{R}} \mathrm{Tr}\,\Big[\frac{d}{dy}\,U(x,y)\,\frac{d}{dy}\,U(y,x)\Big]\,dy\;\;\mathrm{Tr}\,\big[U(x,x)^3\big]\,,
\end{aligned}$$

and similarly

$$\Big(\int_{\mathbb{R}}\Big|\,\mathrm{Tr}\,\Big[U(x,y)\,\frac{d}{dy}\,U(y,x)\,U(x,x)\Big]\Big|\,dy\Big)^2 \leq \int_{\mathbb{R}} \mathrm{Tr}\,\Big[\frac{d}{dy}\,U(x,y)\,\frac{d}{dy}\,U(y,x)\Big]\,dy\;\;\mathrm{Tr}\,\big[U(x,x)^3\big]\,.$$

Thus, using this, and setting $x = z$ in (7), we arrive at

$$\left| \operatorname{Tr} \left[U(x,x)^3 \right] \right| \leq \int_{\mathbb{R}} \operatorname{Tr} \left[\frac{d}{dy} U(x,y) \frac{d}{dy} U(y,x) \right] dy \, .$$

Integrating with respect to x we finally obtain

$$\begin{aligned} &\int_{\mathbb{R}} \left| \operatorname{Tr} \left[U(x,x)^3 \right] \right| dx \\ &\qquad \leq \sum_{n,k=1}^{N} \sum_{i,j=1}^{M} \int_{\mathbb{R}} \int_{\mathbb{R}} \phi_n(x,i) \, \overline{\phi_n'(y,j)} \, \phi_k'(y,j) \, \overline{\phi_k(x,i)} \, dx \, dy \\ &\qquad\qquad = \sum_{n=1}^{N} \sum_{j=1}^{M} \int_{\mathbb{R}} |\phi_n'(x,j)|^2 \, dx \, , \end{aligned}$$

which completes the proof. □

3 Lieb-Thirring inequalities for Schrödinger operators with matrix-valued potentials

Let us assume that $V \in C_0^\infty(\mathbb{R})$, $V \geq 0$, be a $M \times M$ Hermitian matrix-valued potential with entries $\{v_{ij}\}_{i,j=1}^M$. Then the negative spectrum of the Schrödinger operator $H = -\frac{d^2}{dx^2} - V$ in $L^2(\mathbb{R})$ is finite.

Denote by $\{\phi_n\}$ the ortho-normal system of eigen-vector functions corresponding to the eigenvalues $\{\lambda_n\}_{n=1}^N$

$$-\frac{d^2}{dx^2} \phi_n - V \phi_n = \lambda_n \phi_n \, .$$

Clearly,

$$\sum_n \lambda_n = \sum_{n,j} \int_{\mathbb{R}} |\phi_n'(x,j)|^2 \, dx - \operatorname{Tr} \left[\int_{\mathbb{R}} V(x) \, U(x,x) \, dx \right]$$

and by Hölder's inequality for traces,

$$\int_{\mathbb{R}} \operatorname{Tr} \left[V(x) \, U(x,x) \right] \, dx \leq \left(\int_{\mathbb{R}} \operatorname{Tr} \left[V^{3/2}(x) \right] dx \right)^{\frac{2}{3}} \left(\int_{\mathbb{R}} \operatorname{Tr} \left[U(x,x)^3 \right] \, dx \right)^{\frac{1}{3}},$$

so that using Theorem 3

$$\sum_n \lambda_n \geq X - \left(\int_{\mathbb{R}} \operatorname{Tr} \left[V^{3/2}(x) \right] \, dx \right)^{\frac{2}{3}} X^{\frac{1}{3}}$$

with $X := \int_{\mathbb{R}} \mathrm{Tr}\,\left[U(x,x)^3\right] dx$. Minimising the right hand side with respect to X we finally complete the proof of Theorem 1

$$\sum_n \lambda_n \geq -\frac{2}{3\sqrt{3}} \int_{\mathbb{R}} \mathrm{Tr}\,\left[V^{3/2}(x)\right]\, dx\,.$$

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